Manipul8: An Interactive Experience to Inspire Pattern-Based Algebraic Thinking and Representational Fluency

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Abstract
Transitioning students from arithmetic to algebraic thinking is a primary challenge in mathematics education. Visual patterns and physical manipulatives can be helpful, but students often struggle to see the connections between different representations. Manipul8 combines visual patterns with physical manipulatives and provides digital scaffolding to help students develop representational fluency. Using a tabletop tangible user interface, students manipulate equation frames with cutouts for quadratic, linear, and constant terms. Tangible, interchangeable terms are represented either traditionally or as quantities of shapes. The projected digital image provides real-time feedback showing algebraic growth patterns generated from the user-chosen equation structure and terms. Color provides scaffolding for noticing the connections between the equation's terms and visual-pattern-based representations.

Author Keywords
math; algebra; tangible user interfaces; multiple representations; visual patterns; functions; representational fluency

CCS Concepts
•Applied computing → Interactive learning environments;

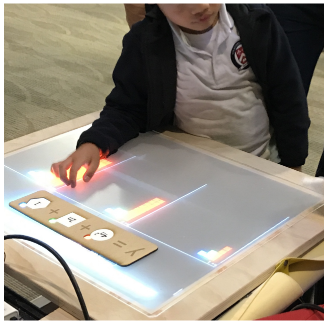

Figure 1: A child explores functions with Manipul8.

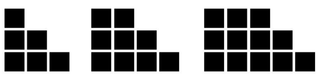

Figure 2: A typical visual growth pattern students consider when beginning to work with algebraic concepts.

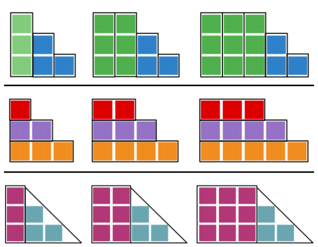

Figure 3: Three ways to visualize a single growth pattern.

Introduction

Understanding algebraic functions is a prerequisite for success in advanced mathematics, physics, computing, and related fields. However, the movement from arithmetic to algebraic thinking, which occurs sometime between late elementary to early high school, is often a significant hurdle for students [2]. We posit that intentional use of visualization and structured embodied interaction may ease the transition and enable students to more fully and naturally realize this type of mathematical thinking.

Visual patterns are a common entry to algebraic thinking [12], as they allow learners to conceptualize algebraic growth in various ways (See Figure 3) [8]. Manipul8 integrates visual patterns, physical manipulatives, and real-time feedback which scaffolds student thinking toward the exploration of and relationships between algebraic concepts.

Background and Related Work

We focus on the development of algebraic intuition and noticing within visual growth patterns. Research has specifically identified the multiple representations and access points for these patterns as rich for the development of student understanding [8, 7] but also problematic in that students struggle to make representational connections between the visual and symbolic forms [5]. Manipul8 builds on: 1. the utility of concrete manipulatives [11] to promote algebraic fluency, 2. the importance of physical manipulation in promoting mathematical understanding and transfer [6], and 3. the clarity of color to aid in the visualization of mathematic ideas [3]. Manipul8 contributes to this scholarship but separates itself from the field in that its dual tangible and digital visualizations allow for the consideration of pattern-based algebraic growth ideas in ways that are concrete, exploratory, yet tightly-coupled; further Manipul8 includes integrated, but fadeable scaffolding, explicitly supporting users in noticing connections between representational forms.

Visual Patterns and Algebraic Thinking

Visual patterns are often an entry point to algebraic reasoning. Pattern questions ask students to generalize growth, generate missing steps, and re-conceptualize growth as functions [12]. Students often encounter difficulties with this pattern work. Typical student weaknesses include: geometric visualization of functions, visualization of additional pattern terms, and formation of conceptual connections between position value ('n') to the pattern itself [5].

Scaffolding for Representational Fluency

Visual representations can allow for "new and deep understandings" [3] when considering patterns as functions. While physical manipulatives can be helpful [11], their use in classrooms does not guarantee student understanding. One common explanation is that students often do not link manipulatives and symbols with representational fluency [4]. Manipul8 supports the development of representational fluency with clear ties between visualizations; the visual patterns dynamically respond as terms are interchanged in an equation frame, and color scaffolding components are included to help users conceptually connect a term to its corresponding portion of the generated pattern. As student algebraic thinking and representational fluency builds, the color scaffolds can fade [10].

Tangibles and Multiple External Representations

Research has shown that Multiple External Representations (MERs) further support the formation of connections between representations, in that learners can view many representations, try multiple strategies, and may understand an unfamiliar representation via its juxtaposition with a familiar form [1]. Further, tangible representations can reduce a learner's cognitive load, appropriately constrain

Representational Fluency
The ability to fluidly translate between and relate representations of the same mathematical idea

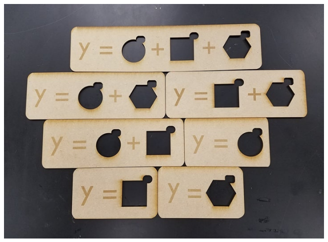

Figure 4: Users have seven equation frame options, representing all combinations of quadratic, linear, and constant forms.

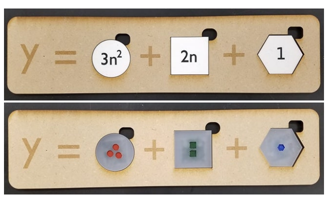

Figure 5: Equation frames have cutouts for interchangeable term manipulatives. Circles are quadratic terms; squares are linear terms; and hexagons are constants. Negative spaces at the upper-right of each manipulative serve as both fingerholds and are digitally backlit with color to provide scaffolding between representations.

user inferences, promote exploratory behavior, and enhance learning relative to multi-touch interfaces. Additionally, tabletop tangible user interface environments allow for tightly-coupled inputs and outputs, as both the input and output can overlap in the same plane [9].

Design

Manipul8 allows students to concretely experience the ideas and necessity of mathematical structure and variable representation using physical manipulatives within a digital platform. Shapes represent different types of algebraic growth, minimizing formal barriers to access. Users first choose an equation structure - a physical frame with negative space(s) that allows for any combinatorial possibilities of quadratic, linear, and constant terms (See Figures 4, 5). Using manipulatives representative of algebraic terms, students fill the frame, generating a digital growth pattern. Learners physically explore math in a fluid way, while experiencing scaffolds, such as color-codes, that support the association of symbolic representations with corresponding portions of the visual pattern (See Figure 6).

Manipulatives and Physical Structure
During user testing we found the tangible components make essential contributions to the fluidity of the exploration. The wooden equation frames are particularly important, as they demand the user to first conceptualize the type of growth an idea requires, choose the appropriate complementary frame, and then select the terms. This physical distinction between mathematical structure and term value shifts the learner's cognitive load from considering the ideas synchronously to considering each separately. In this way, the technology encourages the user to consider distinct subcomponents of the function's structure, an important conceptual process for the making of connections between portions of the equation and their representations.

Scaffolds in Digital and Physical Representations
A common learner issue when building algebraic reasoning with pattern growth problems is difficulty in conceptually mapping the corresponding components of each representation. Manipul8 includes intentional scaffolding, both through digital colorization and physical interchangeability of terms, allowing students to visualize the portion of each pattern generated by individual terms within the equation as well as to transfer these terms to new equation structures. A particularly important affordance of the equation frame structure is the negative space in the upper-right of each cutout (See Figure 5). This feature functions both as a fingerhold and zone for digital color backlighting, bathing a chosen term in the same color as the portion of the pattern it generates (See Figure 6). In this way, students are supported in forming connections between mathematical structure, term, and pattern portions that are transferable throughout the Manipul8 experience.

Learner Agency and Assistance
The dual sets of term manipulatives - shape-based faces (See Figure 7) and traditional term-based faces (See Figure 8) - allow for learner agency and choice in how he/she would like to engage with the mathematic ideas. Additionally, the shapes of the negative cutouts within the equation frames as well as the matching shapes of the term manipulatives allow the user flexibility of exploration while still focusing on mathematically simplified standard forms of quadratic, linear, or constant equations.

Further, the slider for values of n affords the user additional agency in how he/she wishes to explore pattern growth. In the default view, Manipul8 displays the first three patterns (n=1, n=2, and n=3) for a specified equation (See figure 6); however, the user can change the value of the slider to consider the visualizations generated by larger values of n.

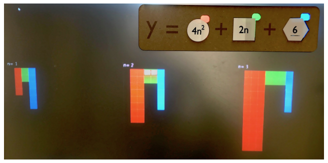

Figure 6: The function's pattern of algebraic growth is shown for three positions.

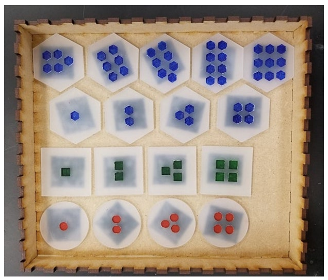

Figure 7: Shape-based faces for term manipulatives allow users to interact without the barrier of algebraic/symbolic notation.

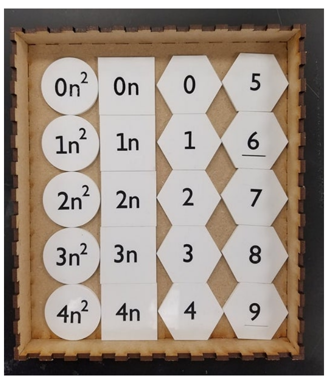

Figure 8: Traditional term-based faces for manipulatives allow Manipul8 to grow with the user's mathematical development.

Conclusion and Future Work

Manipul8 is a tangible user interface allowing students to explore algebraic thinking through pattern analysis and building. Physical manipulatives and the digital projection are tightly-coupled, allowing for scaffolding that meaningfully supports the learner in making connections toward representational fluency. Future empirical work will include analysis of student noticing and sense-making practices as well as several additional modes of co-designed interaction.

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